Announcements

1) Fix in notes for description of a nonintegrable derivative.

Theorem: (Fundamental!)

Suppose $g : [a,b] \to \mathbb{R}$ and $f = g'$. Then f has a generalized Riemann integral and, denoting the integral abusively by

$$\int_a^b f(x)\,dx,$$

$$\boxed{g(x) - g(a) = \int_a^x f(t)\,dt}$$

Proof: Observe that,

for $x \in [a,b]$,

$$g(x) - g(a) = \sum_{i=0}^{n-1} \left(g(x_{i+1}) - g(x_i) \right)$$

for any partition P of

$[a,x]$.

Now, for simplicity, let $x = b$
and let $(P, \{c_i\}_{i=0}^{n-1})$ be a tagged
partition on $[a,b]$.

Then

$$\left| g(b) - g(a) - \sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) \right|$$

$$= \left| \sum_{i=0}^{n-1} \left(g(x_{i+1}) - g(x_i) - f(c_i)(x_{i+1} - x_i) \right) \right|$$

$$\leq \sum_{i=0}^{n-1} \underbrace{\left| g(x_{i+1}) - g(x_i) - f(c_i)(x_{i+1} - x_i) \right|}_{\text{estimate this quantity}}$$

Let $\varepsilon > 0$.

Step 1: (making the gauge)

$\forall\ c \in [a,b],\ \exists\ \delta(c) > 0$

$$\left| \frac{g(x) - g(c)}{x - c} - f(c) \right| < \frac{\varepsilon}{b-a}$$

$\forall\quad 0 < |x - c| < \delta(c)$

This is true since $f = g'$

on $[a,b]$. Remember this

means $\forall\ c \in [a,b]$

$$\lim_{x \to c} \left| \frac{g(x) - g(c)}{x - c} - f(c) \right|$$

$$= 0$$

Step 2: Let $(P, \{c_i\}_{i=0}^{n-1})$ be a $\delta(c)$-fine partition of $[a,b]$. Then

$$|g(x_{i+1}) - g(c_i) - f(c_i)(x_{i+1} - c_i)| < \frac{\varepsilon(x_{i+1} - c_i)}{b-a} \quad \text{and}$$

$$|g(c_i) - g(x_i) - f(c_i)(c_i - x_i)| < \frac{\varepsilon(c_i - x_i)}{b-a}$$

Since P is $\delta(c)$-fine,

$x_{i+1} - x_i < \delta(c_i)$ for

all $0 \leq i \leq n-1$.

Then apply step 1 and

observe $x_{i+1} - c_i < x_{i+1} - x_i$

$< \delta(c_i)$

Since $c_i \in [x_i, x_{i+1}]$ and

Similarly,

$c_i - x_i < \delta(c_i)$.

Applying step 1,

$$\left| \frac{g(x_{i+1}) - g(c_i)}{x_{i+1} - c_i} - f(c_i) \right| < \frac{\varepsilon}{b-a}$$

and

$$\left| \frac{g(c_i) - g(x_i)}{c_i - x_i} - f(c_i) \right| < \frac{\varepsilon}{b-a}.$$

Multiplying through by $x_{i+1} - c_i$ and $c_i - x_i$, respectively, we get step 2.

Using step 2,

$$|g(x_{i+1}) - g(x_i) - f(c_i)(x_{i+1} - x_i))$$

$$= |g(x_{i+1}) - g(c_i) + g(c_i) - g(x_i)$$
$$- f(c_i)(x_{i+1} - c_i + c_i - x_i)|$$

$$\leq |g(x_{i+1}) - g(c_i) - f(c_i)(x_{i+1} - c_i)|$$
$$+ |g(c_i) - g(x_i) - f(c_i)(c_i - x_i)|$$
$$< \frac{\varepsilon}{b-a}(x_{i+1} - c_i) + \frac{\varepsilon}{b-a}(c_i - x_i)$$

$$= \frac{\varepsilon}{(b-a)}(x_{i+1}-x_i)$$

This then implies

$$\left| g(b)-g(a) - \sum_{i=0}^{n-1} f(c_i)(x_{i+1}-x_i) \right|$$

$$< \sum_{i=0}^{n-1} \frac{\varepsilon}{(b-a)}(x_{i+1}-x_i)$$

$$= \frac{\varepsilon}{(b-a)}(b-a) = \varepsilon \quad \square$$

Example: (Dirichlet)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

We know f is not Riemann integrable on $[0,1]$.

We show that f has generalized Riemann integral equal to 0.

We need to construct a gauge. Let $\varepsilon > 0$.

Define $\delta(c) = 1$ for $x \notin \mathbb{Q}$. Let $(x_i)_{i=1}^{\infty}$ be an enumeration of the rationals in $[0,1]$.

Define $\delta(x_i) = \dfrac{\varepsilon}{2^{i+1}}$

Let $(P, \{c_i\}_{i=0}^{n-1})$

be a tagged partition

that is $\delta(c)$-fine.

$$\sum_{i=0}^{n-1} f(c_i)(y_{i+1} - y_i)$$

If $c_i \notin \mathbb{Q}$, $f(c_i) = 0$.

If $c_i \in \mathbb{Q}$, then

$f(c_i) = 1$. Then $y_{i+1} - y_i < \delta(c_i)$

$$= \frac{\varepsilon}{2^k}$$

for some $k \in \mathbb{N}$.

Since c_i is at most in the intervals $[x_i, x_{i+1}]$ and $[x_{i+1}, x_{i+2}]$, $\delta(c_i) = \frac{\varepsilon}{2^k}$ at most twice for any $k \in \mathbb{N}$.

Therefore,

$$\sum_{i=0}^{n-1} f(c_i)(y_{i+1} - y_i) < 2 \sum_{n=2}^{\infty} \frac{\varepsilon}{2^k}$$

$$= \varepsilon.$$

A note on Abbott's construction

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Abbott's $(f_n)_{n=1}^{\infty}$ construction:

$$f_1(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{3}, \ \frac{2}{3} \leq x \leq 1 \\ g(x - 1/3), & x \text{ "a bit to the right" of } 1/3 \\ g(-x + 2/3), & x \text{ "a bit to the left" of } 2/3 \\ \text{Something else}, & \text{remaining } x \end{cases}$$

Where the vague terms are interpreted precisely so that f_1' is continuous except at $1/3$, $2/3$.

Can you do this?

Yes, this is one way:

Let $g_1(x) = g(x - 1/3)$

$g_2(x) = g(-x + 2/3)$

$$x_1 = \frac{1}{3} + \frac{1}{4\pi}$$

$$x_2 = \frac{2}{3} - \frac{1}{4\pi}$$

$$1/3 < x_1 < x_2 < 2/3$$

$$g_1(x_1) = g\left(\frac{1}{4\pi}\right) = 0$$

$$g_2(x_2) = g\left(\frac{1}{4\pi}\right) = 0$$

$$g_1'(x_1) = -1$$

$$g_2'(x_2) = 1.$$

for $x_1 \le x \le x_2$,

define $f_1(x) = \dfrac{2x_1}{e^{x_1^2}} e^{-\left(x - \frac{x_1+x_2}{2}\right)^2}$ ✓

for $\frac{1}{3} \le x < x_1$, $f_1(x) = g_1(x)$, ✓

for $x_2 < x \le 2/3$, $f_1(x) = g_2(x)$.